Linear threshold neurons

Sebastian Seung
What is the function of synaptic convergence?
Special case: binary output

- “linear threshold neuron”
- Heaviside step function

\[ H \left( \sum_j w_j x_j - \theta \right) \]
Special case: binary input

• Suppose also that the $N$ input variables take on binary values
  – $1 = \text{true}$
  – $0 = \text{false}$

• What Boolean functions can be realized by an LT neuron?
George Boole

AN INVESTIGATION
OF
THE LAWS OF THOUGHT,
ON WHICH ARE FOUNDED
THE MATHEMATICAL THEORIES OF LOGIC
AND PROBABILITIES.

BY
GEORGE BOOLE, LL.D.
PROFESSOR OF MATHEMATICS IN QUEENS' COLLEGE, CORK.

LONDON:
WALTON AND MABERLY,
UPPER SOWESTREET, AND IVY-LANE, PATERNOSTER-ROW.
CAMBRIDGE: MACMILLAN AND CO.
1854.
A LOGICAL CALCULUS OF THE
IDEAS IMMANENT IN NERVOUS ACTIVITY

WARREN S. McCulloch and WALTER PITTS

FROM THE UNIVERSITY OF ILLINOIS, COLLEGE OF MEDICINE,
DEPARTMENT OF PSYCHIATRY AT THE ILLINOIS NEUROPSYCHIATRIC INSTITUTE,
AND THE UNIVERSITY OF CHICAGO

Because of the "all-or-none" character of nervous activity, neural
events and the relations among them can be treated by means of proposi-
tional logic. It is found that the behavior of every net can be described
in these terms, with the addition of more complicated logical means for
nets containing circles; and that for any logical expression satisfying
certain conditions, one can find a net behaving in the fashion it describes.
It is shown that many particular choices among possible neurophysiologi-
cal assumptions are equivalent, in the sense that for every net behav-
ing under one assumption, there exists another net which behaves un-
der the other and gives the same results, although perhaps not in the
same time. Various applications of the calculus are discussed.
Excitatory synapses

- Suppose that $w_i = 1$ for all $i$
- It doesn’t matter which inputs are active — only how many are active.

$$H\left(\sum_j x_j - \theta\right)$$
Logical AND

• “conjunction”

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<th>$x_1$</th>
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<th>$x_3$</th>
<th>$x_1 \land x_2 \land x_3$</th>
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Logical OR

- “disjunction”

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\[
x_1 \lor x_2 \lor x_3 \quad \theta = 0.5 \\
\]

\[
\begin{align*}
w_1 &= 1 \\
w_2 &= 1 \\
w_3 &= 1
\end{align*}
\]
Selectivity is controlled by threshold

- **AND**
  - highly selective
  - high threshold

- **OR**
  - indiscriminate
  - low threshold
Inhibition as a dynamic threshold

• Suppose that $w_i = \pm 1$ for all $i$
• Then what matters is the number of active excitatory inputs minus the number of active inhibitory inputs

\[
H\left(\sum_{j=1}^{n} x_j - \sum_{j=n+1}^{N} x_j - \theta\right)
\]
Inhibition as negation

- Nonmonotone conjunction

\[
x_1 \land \bar{x}_2 \land x_3 = H(x_1 + \bar{x}_2 + x_3 - 2.5) \\
= H(x_1 + 1 - x_2 + x_3 - 2.5) \\
= H(x_1 - x_2 + x_3 - 1.5)
\]

inhibitory synapse
Claim:
A conjunction or disjunction of $N$ variables or their negations can be realized by an LT neuron.
Weighted voting model

• Inputs: “Aye” or “Nay”
• Conjunction and disjunction: democratic
• Synaptic strength: some votes are more powerful than others
• Neuron as decisionmaker
  – weigh the evidence
  – compare with a threshold
Vector notation

- weight vector \( \mathbf{w} = (w_1, w_2, \ldots, w_N) \)
- input vector \( \mathbf{x} = (x_1, x_2, \ldots, x_N) \)
- threshold \( \theta \)
- inner/scalar product \( \mathbf{w} \cdot \mathbf{x} = \mathbf{w}^T \mathbf{x} = \sum_{i=1}^{N} w_i x_i \)

\[ H(\mathbf{w} \cdot \mathbf{x} - \theta) \]
Separating hyperplane

- $w$ and $\theta$ define a hyperplane that divides the input space into half-spaces.
- This hyperplane is sometimes called the “decision boundary.”
Linear separability

separable

nonseparable
Boolean functions

AND

OR

XOR

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<th>$x_1$</th>
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<th>AND</th>
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The LT neuron cannot represent XOR.
Preferred stimulus

- Direction in input space along which minimal amplitude is needed for activation.
  - (assume positive threshold)
- Same direction as the weight vector.
An LT neuron “prefers” a stimulus in the same direction as its weight vector.
MNIST database

• yann.lecun.com
Class means

\[a \propto 2 + 2 + 2 + \ldots\]

\[b \propto 3 + 3 + 3 + \ldots\]
Mean vs. difference of means
Summary

• Boolean functions
  – Conjunction and disjunction
  – Weighted voting
• Separating hyperplane
  – Preferred stimulus
• What is the function of synaptic convergence?
Monotonic boolean functions

- Can be constructed from AND and OR
- Can be constructed from LT neurons with excitation only.

\[ a_1 \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n \]

\[ f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n) \]