Multilayer perceptrons

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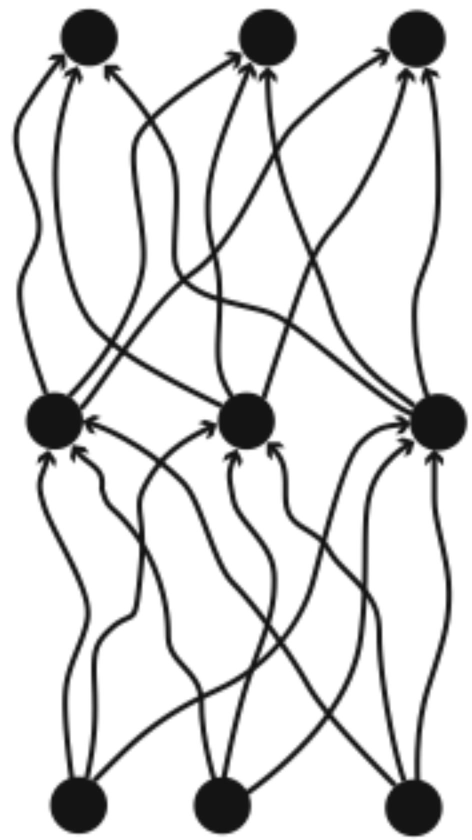
Layered networks

- Two layers of LT neurons
  - (three layers if input neurons are included)
- Two layers of synapses.
- No loops



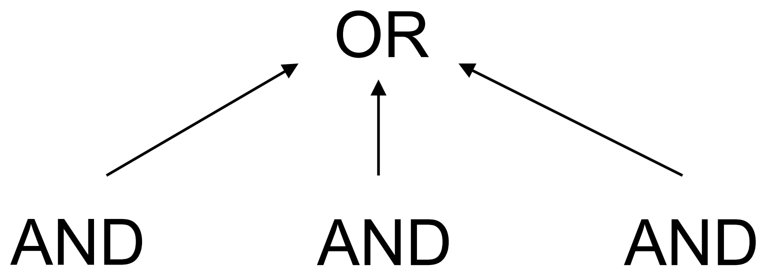
Two questions about representational power

- What can an MLP compute?
  - any boolean function
- What can an MLP compute efficiently?
  - a vanishingly small fraction of boolean functions

Claim:
Any Boolean function can be computed by a perceptron with two layers of synapses.

Disjunctive normal form (DNF)

- Any boolean function can be written in disjunctive normal form.
- Disjunction of conjunctions



DNF construction

x_1	x_2	x_3	$f(x_1,x_2,x_3)$	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	0	
1	0	0	0	
1	0	1	1	$\longrightarrow x_1 \wedge \bar{x}_2 \wedge x_3$
1	1	0	1	$\longrightarrow x_1 \wedge x_2 \wedge \bar{x}_3$
1	1	1	1	$\longrightarrow x_1 \wedge x_2 \wedge x_3$

$$f = (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3)$$
$$= x_1 \wedge (x_2 \vee x_3)$$

Any DNF can be written as a 2-layer perceptron

- AND of variables or their negations
  - LT neuron with excitatory and inhibitory synapses.
- OR
  - LT neuron with excitatory synapses and low threshold.

What’s the catch?

- The number of conjunctions required may be exponentially large.
- I.e., there is no guarantee that the perceptron representation is efficient.

Efficiency

- Number of synapses
- Serial computer
  - Time
- Parallel computer
  - Space
  - Energy

Claim:
Most boolean functions require a perceptron of exponential size.

How many boolean functions can be realized by an MLP with S synapses?

- How many MLPs are there?

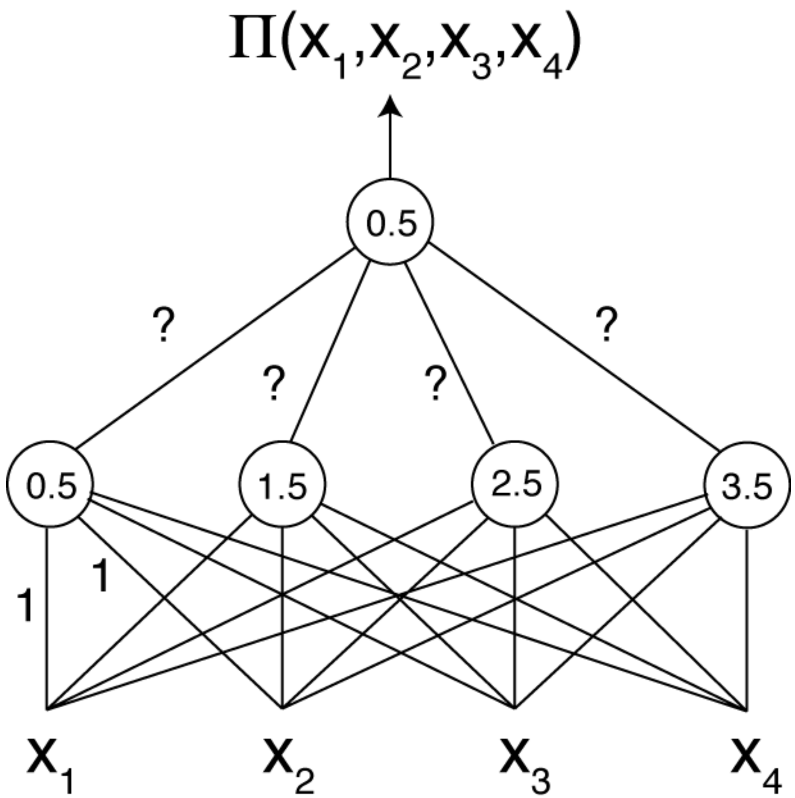
- How many functions can be realized by each MLP?

Parity function

- generalization of XOR to *N* variables
- addition modulo 2

$$\Pi(x_1, \ldots, x_N) = \begin{cases} 1, & \text{if } \sum_{i=1}^{N} x_i \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

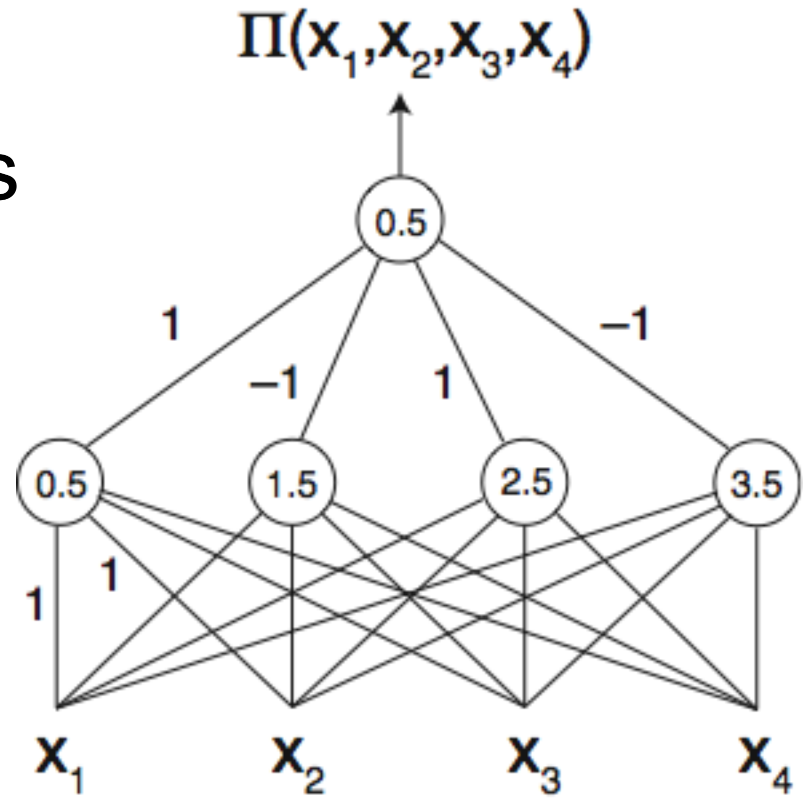
Which 2nd layer weights implement parity?



- A: –1,1,–1,1
- B: 1,–1,1,–1
- C: 1,2,3,4
- D: 1,2,4,8

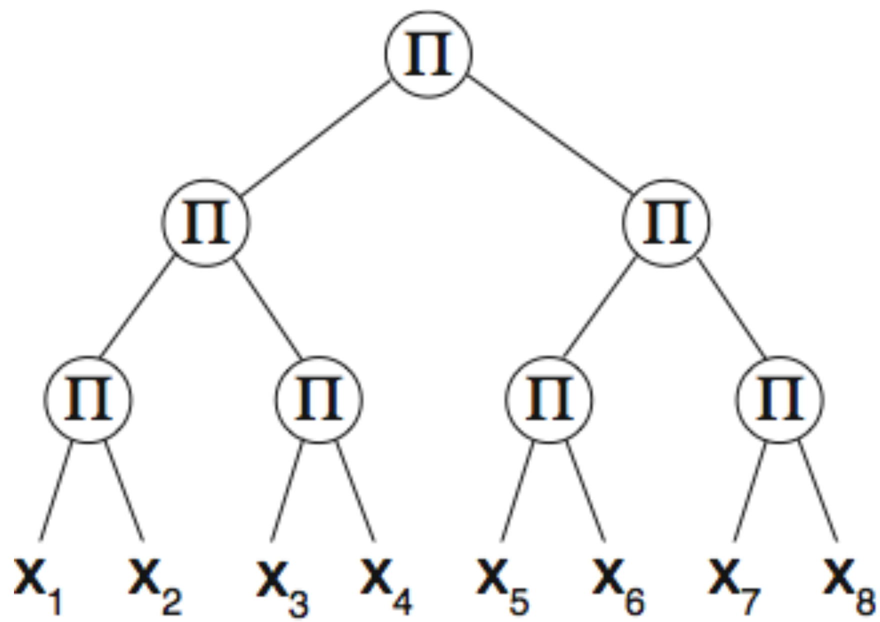
Parity as an MLP with 2 layers

- O(N) neurons
- $O(N^2)$ synapses



Parity as an MLP with log N layers

- O(N) neurons
- O(N) synapses



Depth-size tradeoff

- A deeper network may be able to realize a computation with fewer neurons or synapses.

Circuit complexity theory

- What can circuits of logic gates compute?
- How large a circuit is required?
- Branch of computational complexity theory.

Nonlinearity is essential for computational power

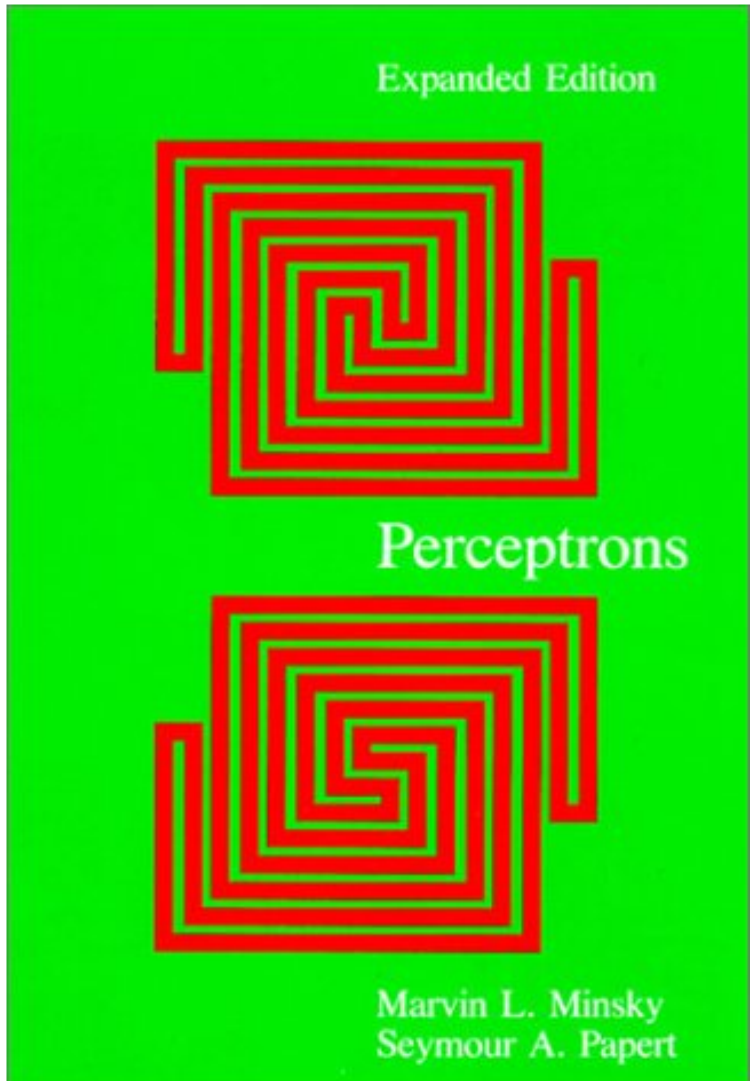
- Increasing the depth of a perceptron makes no difference if the activation function is linear.

Perceptrons can also be used to compute analog functions

For example, use a sigmoidal activation function.

Any smooth function can be approximated by an MLP with two layers of synapses and a sigmoidal activation function.

Connectedness



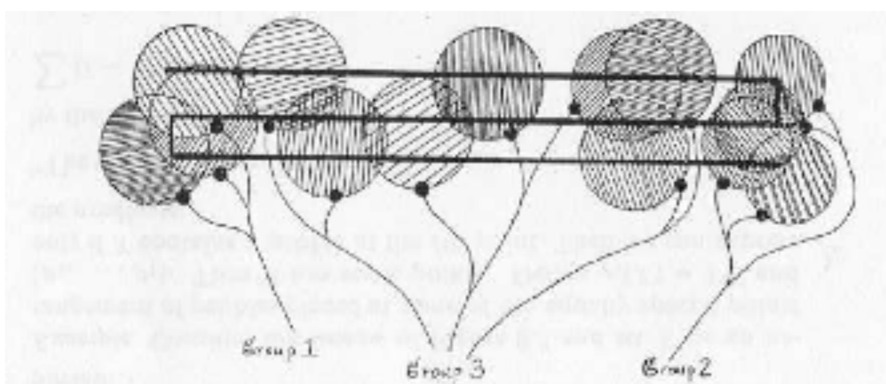
Minsky-Papert definition

- Requires only that the final step in the computation is an LT neuron.

$$H\left(\sum_{a} w_a \varphi_a(\mathbf{x})\right)$$

Diameter-limited perceptrons

- Assume that the input vector is organized as a 2d image.
- Each φ depends on a set of pixels with a limited diameter.



Theorem

No diameter-limited perceptron can compute connectedness.

Proof

- Reduce to XOR